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DIFFERENTIAL-GEOMETRIC CONSIDERATIONS ON THE HODOGRAPH TRANSFORMATION FOR IRROTATIONAL CONICAL FLOW

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ABSTRACT

Local properties of the hodograph transformation for irrotational conical flow, as derived by means of differential geometry, are presented. The supersonic flow over the expansion side of a swept forward arrow wing or delta wing, both with supersonic leading edges, is discussed as an example of how knowledge of these properties, especially in singular points of the transformation, can be employed for the formulation of boundary value problems.

INTRODUCTION

This paper discusses supersonic spatial flows which are conical in the sense originally introduced into aerodynamics by Busemann.[1] In such a flow the velocity and the conditions defining the state of the gas, e.g., the pressure and temperature, are constant on rays through one point of the physical space, called the center of the conical field.

Since the equations for conical flow are nonlinear, an analytical treatment of the problem has only been given by linearization of the equations, which was initiated by Busemann,[2] whereas also higher order approximations were considered. In the nonlinear theory, the flow around a specific body is obtained as a numerical solution of the differential equations. It is of interest for the proper formulation of boundary value problems to consider these equations in more detail. This has been done to some extent by Bulakh in a number of papers (Refs. 3, 4, and papers cited therein). The present paper presents a discussion on the local properties of the hodograph transformation for irrotational conical flow. These properties are being used to express quantities of physical interest in terms of the geometry of the hodograph surface. A study of the hodograph surface then gives information on the structure of solutions of the equations for conical flow. Furthermore, an example will be given how these properties can be employed in the formulation of boundary value problems. Because of timt limitations this discussion will be of a condensed nature. A more detailed accoune has been given in Refs. 5 and 6.

ANALYSIS OF IRROTATIONAL CONICAL FLOW ON THE UNIT SPHERE

In the physical space let a right-handed coordinate system x, y, z be fixed with the origin at the center of the conical field, and let u, v, and w be the components of the velocity along these axes, respectively. The coefficients of viscosity and heat conduction of the gas are assumed to be zero and the gas is assumed to be perfect. If the flow is free of rotation, a conical velocity potential F may be defined, related to the velocity potential for three-dimensional flow φ by

$$\varphi(x,y,z) = zF(\xi,\eta) \tag{1}$$

where

$$\xi = \frac{x}{z} \text{ and } \eta = \frac{y}{z} \tag{2}$$

The velocity components then become

$$u = F_\xi,\ v = F_\eta, \text{ and } w = F - \xi F_\xi - \eta F_\eta \tag{3}$$

If it is assumed, moreover, that the flow is isentropic, the usual conservation laws yield the following equation

$$F_{\xi\xi}\left[1 + \xi^2 - \frac{(u - w\xi)^2}{a^2}\right] + 2F_{\xi\eta}\left[\xi\eta - \frac{(u - w\xi)(v - w\eta)}{a^2}\right] + F_{\eta\eta}\left[1 + \eta^2 - \frac{(v - w\eta)^2}{a^2}\right] = 0 \tag{4}$$

where a is the local speed of sound, related to the velocity components by

$$a^2 = \frac{\gamma + 1}{2} a_*^2 - \frac{\gamma - 1}{2}(u^2 + v^2 + w^2) \tag{5}$$

a_* is the critical speed of sound, and γ is the ratio of specific heats

$$\left(\gamma = \frac{C_p}{C_v}\right)$$

The equation for the characteristic directions of this quasi-linear differential equation can be written as:

$$\left(\frac{d\eta}{d\xi}\right)^2_{\text{char}}\left[1 + \xi^2 - \frac{(u - w\xi)^2}{a^2}\right] - 2\left(\frac{d\eta}{d\xi}\right)_{\text{char}}\left[\xi\eta - \frac{(u - w\xi)(v - w\eta)}{a^2}\right] + \left[1 + \eta^2 - \frac{(v - w\eta)^2}{a^2}\right] = 0 \tag{6}$$

Since the flow is depending on ξ and η only, the flow can be described on a sphere with unitary radius and center in the center of the conical flow field. The characteristics found from Eq. (6) may then be drawn on the unit sphere and called *conical characteristics.* Also, *conical streamlines* may be defined on the unit sphere, the conical streamline being the intersection curve with the sphere of a conical stream surface with apex at the center of the conical flow. In an arbitrary point,—and without loss of generality, this point may be taken to coincide with $\xi = \eta = 0$, the local value of the characteristic directions may be written, with Eq. (6), as

$$\left(\frac{dH}{d\Xi}\right)_{\text{char}} = \frac{\pm 1}{\sqrt{\dfrac{U^2}{a^2} - 1}} \tag{7}$$

where use has been made of a rotated coordinate system, such that the ξ axis is parallel to the velocity component normal to the radius. The rotated system will be indicated by capital letters. The conical characteristics thus subtend the Mach angle, defined in terms of the velocity component normal to the radius, with the conical streamline. Let us call this Mach angle the *conical Mach angle* μ_C and the Mach number, defined in terms of the velocity component U, the *conical Mach number* M_C ($= U/a$), then Eq. (7) can be written as

$$\left(\frac{dH}{d\Xi}\right)_{\text{char}} = \frac{\pm 1}{\sqrt{M_c{}^2 - 1}} = \pm \tan \mu_c \tag{8}$$

The conical characteristic directions are real and have two different values for $M_C > 1$; the equation is then of hyperbolic type, and the flow will be called *conical-supersonic flow.* For $M_C = 1$ the two conical characteristic directions are coincident, real and perpendicular to the conical streamline; the equation is parabolic, and the flow will be termed *conical-sonic.* If $M_C < 1$, the conical characteristic directions are imaginary; the equation is of elliptic type, and the flow will be called *conical-subsonic* flow. Points of the unit sphere where $U = 0$ will be called *conical stagnation points.*

In order to illustrate the quantities defined on the unit sphere, in Fig. 1 parallel flow is given as an example of conical flow.

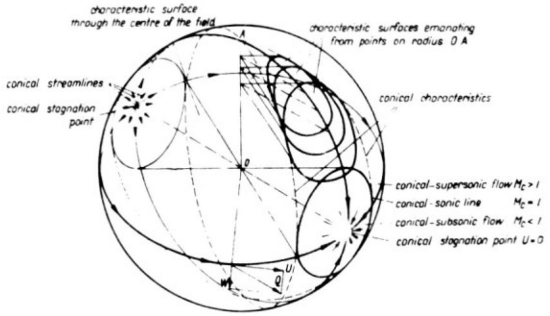

Fig. 1. Description on the unit sphere of parallel flow throughout physical space.

HODOGRAPH TRANSFORMATION FOR IRROTATIONAL CONICAL FLOW

The hodograph transformation for irrotational conical flow is obtained in the usual way by introducing the Legendre potential

$$\chi(u,v) = u\xi + v\eta - F(\xi,\eta) \tag{9}$$

which compared with Eq. (3) shows that

$$\chi(u,v) = - w\ (u,v) \tag{10}$$

and also that

$$\chi_u = - w_u = \xi \qquad \text{and} \qquad \chi_v = - w_v = \eta \tag{11}$$

Differentiation of Eq. (11) yields

$$\begin{aligned} d\xi &= - dw_u = - (w_{uu}du + w_{uv}\, dv) \\ d\eta &= - dw_v = - (w_{vu}du + w_{vv}\, dv) \end{aligned} \tag{12}$$

If the Jacobian determinant $\Delta \equiv w_{uu}w_{vv} - w_{uv}^2$ is finite and different from zero Eq. (12) may be solved for du and dv, and the transformation is locally one-to-one. Singularities in the transformation occur for $\Delta = 0$ and $\Delta \rightarrow \infty$.

The differential equation in the hodograph space becomes

$$w_{uu}\left[1 + w_v^{\ 2} - \frac{(v + ww_v)^2}{a^2}\right] - 2w_{uv}\left[w_uw_v - \frac{(u + ww_u)\ (v + ww_v)}{a^2}\right] + w_{vv}\left[1 + w_u^{\ 2} - \frac{(u + ww_u)^2}{a^2}\right] = 0 \tag{13}$$

and the corresponding equation for the hodograph characteristic directions

$$\left(\frac{dv}{du}\right)_{\text{char}}^2\left[1 + w_v^2 - \frac{(v + ww_v)^2}{a^2}\right] + 2\left(\frac{dv}{du}\right)_{\text{char}}\left[w_uw_v - \frac{(u + ww_u)\ (v + ww_v)}{a^2}\right] + \left[1 + w_u^2 - \frac{(u + ww_u)^2}{a^2}\right] = 0 \tag{14}$$

which reduces in the rotated coordinate system to

$$\left(\frac{dV}{dU}\right)_{\text{char}} = \pm \sqrt{M_c^{\ 2} - 1} \tag{15}$$

THE USAGE OF LOCAL PROPERTIES OF THE TRANSFORMATION TO EXPRESS QUANTITIES OF PHYSICAL INTEREST IN TERMS OF THE GEOMETRY OF THE HODOGRAPH SURFACE

From Eq. (11) it follows that the radius in the physical space is perpendicular to the surface element at the corresponding point on the hodograph surface. The sphere obtained by collecting at one point the unit vectors along the normals to the hodograph surface is the unit sphere in the physical space, as mentioned before. From differential geometry the transformation from the hodograph space to the physical space may be recognized as being the spherical or Gaussian transformation of the hodograph surface. Differential geometry may thus well be used to derive the local properties of the hodograph transformation.

In Ref. 5 a discussion is given of properties related to the curvature of the hodograph surface, expressed by the total or Gaussian curvature

$$K_G = \frac{1}{\rho_1 \rho_2} \tag{16}$$

which equals the Jacobian determinant Δ. In this equation and throughout the rest of this paper, ρ_1 is the major principal radius of curvature and ρ_2 the minor principal radius of curvature. It may be shown, for example, that the $+$ $(-)$ conical characteristic and the $+$ $(-)$ hodograph characteristic map onto each other in such a way that the $+$ $(-)$ conical characteristic is perpendicular to the $-$ $(+)$ hodograph characteristic.

Use of local properties of the transformation makes it possible to express quantities of physical interest in terms of the geometry of the hodograph surface. Of interest, for example, are the magnitude and direction of the acceleration. The component of the acceleration along the streamline may be expressed as

$$g_s = \frac{a}{Mr} (\rho_1 + \rho_2) \tag{17}$$

where r is the radius to the considered point. The component of the acceleration normal to the streamline may be written as

$$g_n = \frac{a}{Mr} |\rho_1 + \rho_2| \sqrt{M^2 \left[1 - M_c^2 \frac{\frac{\rho_2}{\rho_1}}{\left(1 + \frac{\rho_2}{\rho_1}\right)^2} \right] - 1} \tag{18}$$

whereas the angle β which the acceleration makes with the conical streamline may be obtained from

$$\tan \beta = \pm \sqrt{\frac{\left[1 - \frac{\rho_2}{\rho_1} (M_c^2 - 1)\right] \left[M_c^2 - 1 - \frac{\rho_2}{\rho_1}\right]}{1 + \frac{\rho_2}{\rho_1}}} \tag{19}$$

Here, β is measured positive in counterclockwise direction, and without loss of generality β may be taken such that $-\pi/2 \leqq \beta \leqq \pi/2$. By definition, the hodograph streamline makes the same angle with the U-axis. The function given by Eq. (19) is illustrated in Fig. 2. From it, it may be seen when the transformation is regular ($\rho_2/\rho_1 \neq 0$) that for conical-subsonic ($M_c < 1$) and conical-sonic flow ($M_c = 1$), the points on the hodograph surface are hyperbolic points since $\rho_2/\rho_1 < 0$ and $K_G \equiv \Delta < 0$. Conical-supersonic flow ($M_c > 1$) is either mapped

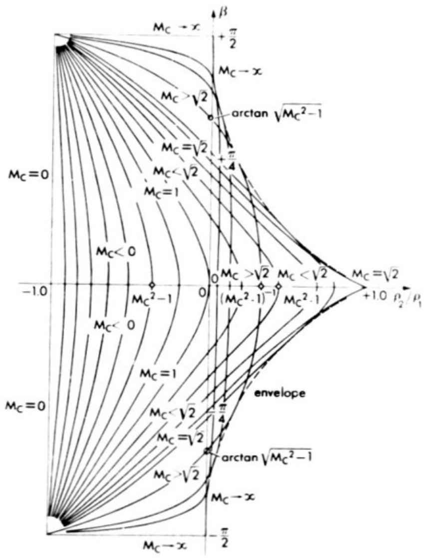

Fig. 2. Direction of the hodograph streamline.

onto hyperbolic or elliptic points, since $\rho_2/\rho_1 < 0$ or $\rho_2/\rho_1 > 0$ and $K_G \equiv \Delta$ either is negative or positive. Furthermore, it can be seen that for elliptic points the hodograph streamline lies in the region which the hodograph characteristics inclose around the U-axis, whereas for hyperbolic points the hodograph streamline lies outside of this region.

If $\rho_2/\rho_1 = 0$, the hodograph streamline is tangent to one of the hodograph characteristics and the transformation is singular. Singularities thus occur for $M_c \geqq 1$ and the flow then has a wavelike structure. It is of interest, therefore, to consider the relation between the strengths of the waves, propagating along the conical characteristics and the shape of the hodograph surface. Classification of the shape of the hodograph surface, as permitted by the differential equation [Eq. (13)], then results in a description of the behavior of the waves, which thereby reveals the structure of conical flow.

RELATION BETWEEN THE STRENGTH OF THE WAVES, PROPAGATING ALONG THE CONICAL CHARACTERISTICS AND THE SHAPE OF THE HODOGRAPH SURFACE

Consider a solution of the quasi-linear equation for irrotational conical flow [Eq. (4)] in a region, where the flow is conical-supersonic, and let PQ be an elementary arc ds of a conical streamline in such a region, as sketched in Fig. 3 Draw the conical characteristics PQ_+ and PQ_- through P and QQ_+ and QQ_- through Q. The change in velocity of a particle when it moves from P to Q, dq, can be shown to consist of two separate contributions, induced by two stationary waves. One of them, dq_+, is completely determined by conditions on the characteristic segment PQ_- and the other, dq_-, is completely determined by conditions on the characteristic segment PQ_+. If the flow is thought to be the result of a construction process, developing in time, dq_+ (dq_-) may be thought to be induced by the motion of a wave, the front of which coincides with characteristic PQ_- ($\mathbf{PQ}_+$), when a given particle is in P and coincides with characteristic Q_+Q (Q_-Q) when this particle has arrived in Q. The speed of propagation of the wave front is then the speed of sound, being equal to the magnitude of the velocity component normal to the wave front. In this sense, it is said that a conical characteristic carries an elementary contribution to the change in velocity along the conical streamline and that this contribution is carried downstream; this means, that changes along the streamline are prescribed by conditions on the characteristics that border the Mach quadrangle upstream.

A simple derivation shows that

$$dq_+ = \tfrac{1}{2}\left(1 + \tan\beta \tan\mu_c\right)\frac{\delta q}{\delta s}\,ds = s_+\,ds \tag{20}$$

$$dq_- = \tfrac{1}{2}\left(1 - \tan\beta \tan\mu_c\right)\frac{\delta q}{\delta s}\,ds = s_-\,ds \tag{21}$$

where $\delta q/\delta s$ is the velocity gradient along the conical streamline. Let us call s_+ and s_- the strength of the $+$wave and $-$wave, respectively, since these quantities are a measure of the elementary contributions, propagating along the $-$ and $+$ characteristics, respectively. If $s > 0$, the speed increases as a result of the contribution, and because in an isentropic flow the pressure then decreases, the wave will be called an expansion wave. Similarly, if $s < 0$ the wave will be called a compression wave. It may be useful further to use the expressions *partially expanding flow* for an expanding flow in which the expansion wave along one characteristic is stronger than the compression wave along the other characteristic and *fully expanding flow* if both waves are expansion waves. Similarly, we might distinguish *partially compressing* and *fully compressing flow*.

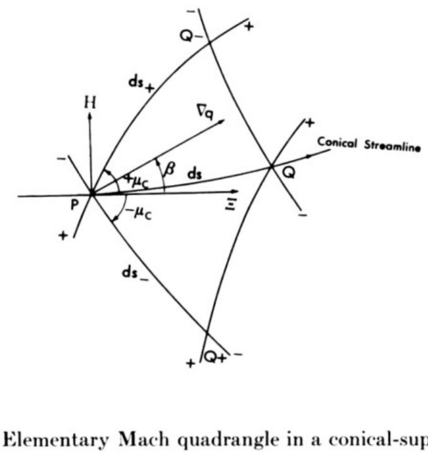

Fig. 3. Elementary Mach quadrangle in a conical-supersonic flow.

The relations obtained for β [Eq. (19)], μ_c [Eq. (8)] and the acceleration Equation (17) may now be used to express the wave strengths in terms of the shape of the hodograph surface. Thence follows:

$$s_+ = \frac{\rho_1}{2MM_c}\left\{1 + \frac{\rho_2}{\rho_1} \pm \sqrt{\frac{\left[1 - \frac{\rho_2}{\rho_1}(M_c^2 - 1)\right]\left[M_c^2 - 1 - \frac{\rho_2}{\rho_1}\right]}{M_c^2 - 1}}\right\} \quad (22)$$

and

$$s_- = \frac{\rho_2}{2MM_c}\left\{1 + \frac{\rho_2}{\rho_1} \pm \sqrt{\frac{\left[1 - \frac{\rho_2}{\rho_1}(M_c^2 - 1)\right]\left[M_c^2 - 1 - \frac{\rho_2}{\rho_1}\right]}{M_c^2 - 1}}\right\} \quad (23)$$

where the upper sign should be taken if $\beta > 0$ and the lower sign if $\beta < 0$. Furthermore, a multiplication of Eqs. (22) and (23) gives

$$s_+s_- = \frac{1}{4M^2}\frac{M_c^2}{M_c^2 - 1}\frac{1}{K_G} \quad (24)$$

It may be seen, by using in Eqs. (20) and (21) the aforementioned property, that for $\rho_2/\rho_1 < 0$ there is $|\beta| > \pi/2\text{-}\mu_c$ (and $|\beta| \leqq \pi/2$ because of the sign convention), whereas for $\rho_2/\rho_1 > 0$ there is $|\beta| < \pi/2\text{-}\mu_c$, or directly from Eq. (24), that *fully expanding* or *fully compressing* flow is mapped onto an *elliptic* point and *partially expanding* or *partially compressing* flow is mapped onto a *hyperbolic* point. If the flow is neither expanding nor compressing because the contributions of the two waves cancel, the point is mapped onto an orthogonal hyperbolic point ($\rho_2/\rho_1 = -1$).

The conical Mach numbers, bounding the scale of conical-supersonic Mach numbers, namely $M_c = 1$ and $M_c \to \infty$, are of interest, because they can occur at natural bounds of regions of conical-supersonic flow. In order to get a better understanding of the structure of the flow, boundaries given by body surfaces are not considered as natural bounds, since the flow may be thought to be extended analytically through the body surface. It can be shown from Eqs. (22) and (23), that at a point of a *conical-sonic line* ($M_c = 1$, $\rho_2/\rho_1 < 0$) only an *expansion wave can arrive*, which is *reflected as a compression wave*, whereas the wave strength goes to infinity (Fig. 4). At a *vacuum point* ($M_c \to \infty$, $\rho_2/\rho_1 < 0$) only a *compression wave can arrive*, which is *reflected as an expansion wave*, both waves having a wave strength equal to zero at the vacuum point (Fig. 5).

In the discussion thus far, it was assumed that the Gaussian curvature is continuous, so that no discontinuities in the second derivatives are present along the characteristics. Let us assume now, that across a hodograph characteristic there is a jump in the Gaussian curvature. It may then be derived from the fact, that the normal curvature of such a characteristic is independent of the side of approach, that the *product of the Gaussian curvature and the strength of the wave*, along the characteristic across which the jump occurs, is an *invariant*. From Eq. (24) then follows directly that the strength of the wave along the other characteristic is *unaffected* by the jump in Gaussian curvature.

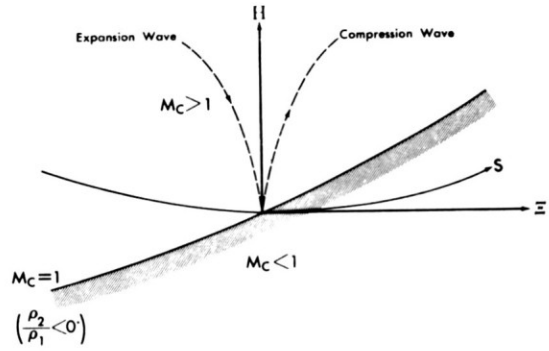

Fig. 4. Reflection of an expansion wave at a conical-sonic line.

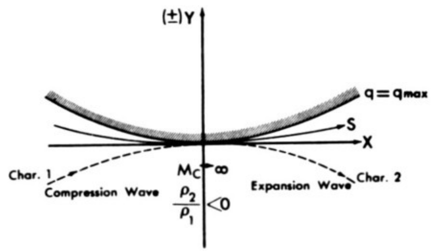

Fig. 5. Reflection of a compression wave at a vacuum point.

SINGULARITIES OF THE TRANSFORMATION

SINGULARITIES CONNECTED WITH A CONTINUOUS GAUSSIAN CURVATURE

Conical Limit Lines of the First Type

A conical limit line of the first type is a line on the hodograph surface where $K_G = 0$ and separates regions with a different sign for K_G. Its image on the unit sphere will also be called a conical limit line. Since $\rho_1 \to \infty$, the acceleration becomes infinite. Furthermore, it can be shown that on the unit sphere, the conical limit line borders a doubly covered region, as a result of which the conical streamlines and one family of characteristics reflect at the limit line from one sheet into the other. The other family of characteristics forms an envelope, which coincides with the limit line. In Fig. 6 local properties of the transformation at a point of a conical limit line of the first type are illustrated for expanding flow.

It can be shown that for expanding flow a *point on the limit line generates expansion waves* along those characteristics in both sheets, which are tangent to the limit line and for compressing flow such *a point absorbs compression waves* traveling along these characteristics. These waves have infinite strength at the limit line. The waves traveling along the *other* family of characteristics are *reflected* into the other sheeet with *equal* and *opposite* strength. In a conical simple wave flow these waves have zero strength and the characteristics forming the envelope are the straight characteristics.

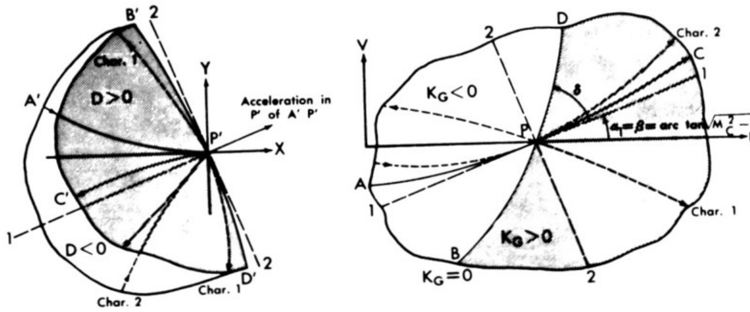

Fig. 6. Point on a conical limit line of the first type in an expanding flow.

A cusp in a conical limit line occurs when the hodograph streamline and hodograph limit line are tangent. If the hodograph limit line coincides with a hodograph streamline, then the conical limit line on the unit sphere is degenerated into a point, wherein the velocity is multivalued. The elementary contributions generated or absorbed by the limit line then spread in an infinite number of directions. Such a centerpoint of waves occurs, for example, at a sharp edge such as a subsonic leading edge ($\rho_2 \neq 0$) or a supersonic leading edge ($\rho_2 = 0$) of a delta wing.

Conical Branch Lines of the First Type

In order to give a definition of a conical branch line it is more convenient to use the Jacobian $D = \delta(u,v)/\delta(\xi,\eta)$ of the inverse transformation, which equals the inverse of the Gaussian curvature K_G^{-1}. A conical branch line of the first type will now be defined as a line on the unit sphere where $D = 0$, and separates regions which have a different sign for D. On the image of the branch line in the hodograph space the Gaussian curvature $K_G \rightarrow \infty$, ($\rho_2 \rightarrow 0$). Both on the unit sphere and in the hodograph space the branch line coincides with a characteristic. The hodograph surface exhibits a fold along the conical branch line; in other words, two sheets of the hodograph surface are tangent to each other along a common edge, which coincides with the branch line. In the hodograph space the branch line is an envelope of the hodograph streamlines. Local properties of the transformation at a point of a conical branch line of the first type for expanding flow are illustrated in Fig. 7.

It can be shown from Eqs. (22) and (23) that the conical *branch line* of the first type is a characteristic, which *does not carry elementary contributions and separates the characteristics of its family, which carry expansion waves from those which carry compression waves.* The other characteristic carries an expansion or a compression wave depending on whether the flow is expanding or compressing at the branch line.

SINGULARITIES CONNECTED WITH A DISCONTINUOUS GAUSSIAN CURVATURE

Conical Limit Lines of the Second Type

If along a hodograph characteristic a region of hyperbolic points on one side of the characteristic is connected with a region of elliptic points on the other side, such that the Gaussian curvature is discontinuous across this characteristic, we will call this line a conical limit line of the second type. As for a limit line of the first type a doubly covered region on the unit sphere appears and reversal of the conical streamlines occurs at the limit line. Local properties of the transformation at a point of a limit line of the second type in expanding flow are illustrated in Fig. 8.

It can be shown that for *expanding flow* the characteristics of the *family to which the limit line belongs carry expansion waves*, traveling in one direction in one sheet and the opposite direction in the other sheet, whereas for *compressing flow* these characteristics carry *compression waves*. As is the case for the acceleration at the limit line, the strength of the wave along the limit line remains finite and jumps across the limit line. Like at a conical limit line of the first type, it can be shown that the waves traveling along the *other* family of characteristics *reflect* from one sheet into the other with *equal* and *opposite* strength.

Conical Branch Lines of the Second Type

A conical branch line of the second type will be defined as a conical characteristic on the unit sphere, which is a dividing line between a region of points where $D > 0$ on one side and a region where $D < 0$ on the other side of the line, such that across the characteristic a discontinuity in D occurs. As for a conical branch line of the first type, it can be shown that in the hodograph space the

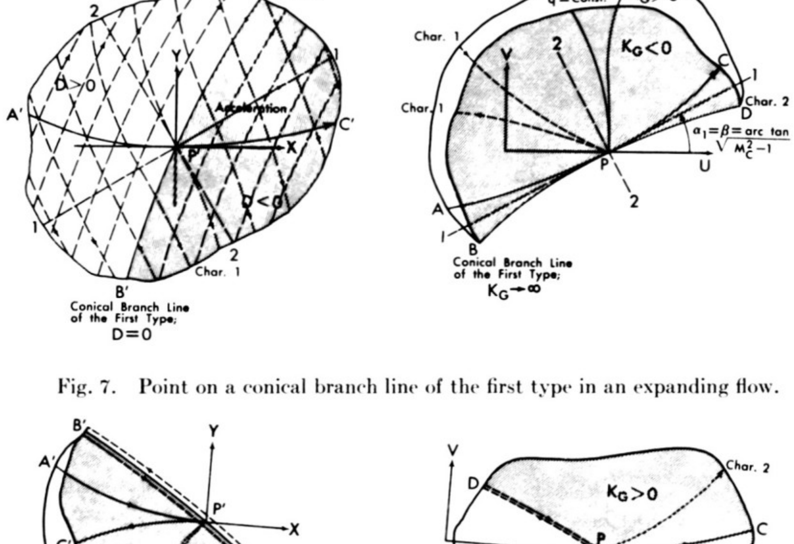

Fig. 7. Point on a conical branch line of the first type in an expanding flow.

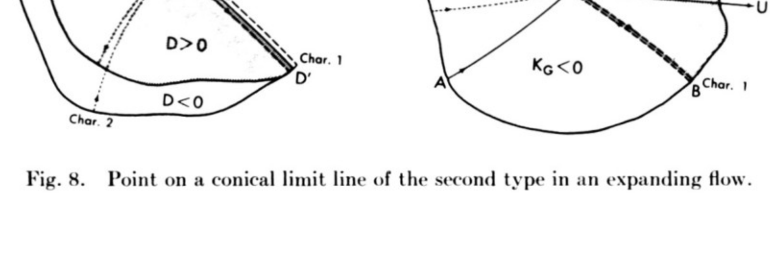

Fig. 8. Point on a conical limit line of the second type in an expanding flow.

branch line is a characteristic, which is the line of tangency of two sheets of the hodograph surface, lying on the same side of the branch line. Across this characteristic the Gaussian curvature is discontinuous and jumps through zero. In Fig. 9 local properties of the transformation at a point of a conical branch line of the second type in expanding flow are illustrated.

It can be shown that the conical *branch line* of the second type is the one characteristic of its family, which *separates those characteristics, which carry expansion waves from those which carry compression waves.* The strength of the wave along the branch line jumps through zero across this line. The strength of the waves along the other family of characteristics is not affected by the discontinuity across the branch line.

Conical Simple Wave Flow

A region of conical simple wave flow may be thought to be obtained as a result of a limiting process, by introducing discontinuities in the Gaussian curvature across two hodograph characteristics C_1 and C_2 of the same family, such that the region in between these characteristics shrinks till it coincides with a curve, while in every point of the region, called an edge surface, $\rho_2 \rightarrow 0$; thus $K_G \rightarrow \infty$ (Fig. 10). The hodograph streamline then becomes tangent to one of the hodograph characteristics as is illustrated in Fig. 2, where lines of constant M_c intersect the line $\rho_2/\rho_1 = 0$ (β axis) at the characteristic angle.

It may be concluded from the invariance of the product of Gaussian curvature and wave strength along a characteristic across which a discontinuity in Gaussian

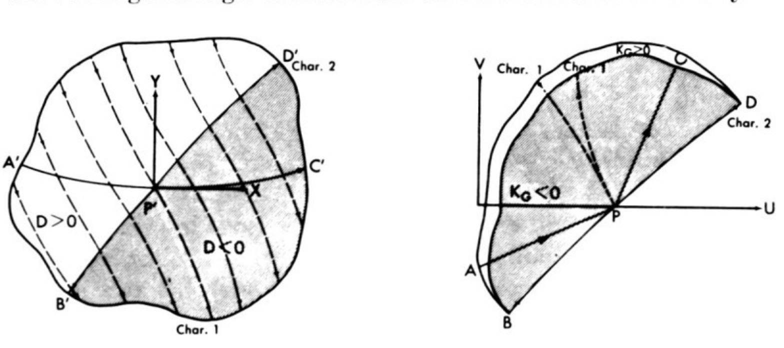

Fig. 9. Point on a conical branch line of the second type in an expanding flow.

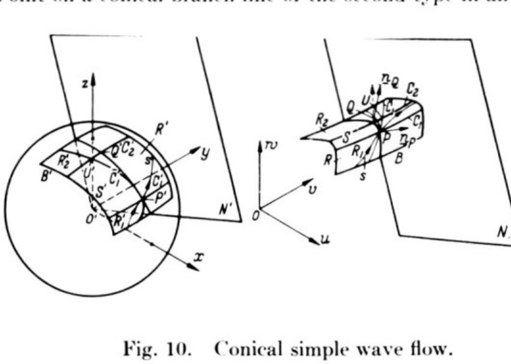

Fig. 10. Conical simple wave flow.

curvature occurs, if $K_G \to \infty$ on the edge surface that the *wave strength* of the waves *along the curved characteristics vanishes.* The straight characteristics carry waves with a strength unequal to zero ($s = \rho_1/MM_c$).

The occurrence of a limit line of the first type in a conical simple wave flow is discussed before as was the situation where the straight characteristics are centered in one point.

Regions of Parallel Flow in a Conical Flow Field

A region of parallel flow may be obtained by extending the limiting process used to achieve conical simple wave flow further by decreasing the strengths of the waves, propagating along the straight characteristics. Since in a conical simple wave flow $s = \rho_1/MM_c$ or $s = \rho/MM_c \sin \Theta$, zero wave strength may only be obtained by letting the radius of curvature of the space curve, representing the simple wave flow, approach to zero in every point of it. As a result, the edge surface shrinks until it coincides with a point and a point surface is obtained. The parallel flow may either be bounded by a straight characteristic (and then the adjacent flow field is a conical simple wave flow) or a conical-sonic line in which case the possible adjacent flow fields may be derived by means of a series expansion and are given in Fig. 11.

If the flow adjacent to a parallel flow along a conical-sonic line is conical-supersonic, it can be shown that a point on the conical-sonic line *generates compression waves along both characteristics* if the flow is compressing and *absorbs expansion waves along both characteristics* if the flow is expanding. This property makes the conical-sonic line, bounding a parallel flow, the counterpart of the limit line of the first type. As a further difference with the limit line, it may be noted that the *wave strengths* at the conical-sonic line are *finite*, in general, instead of infinite as for the case of the limit line.

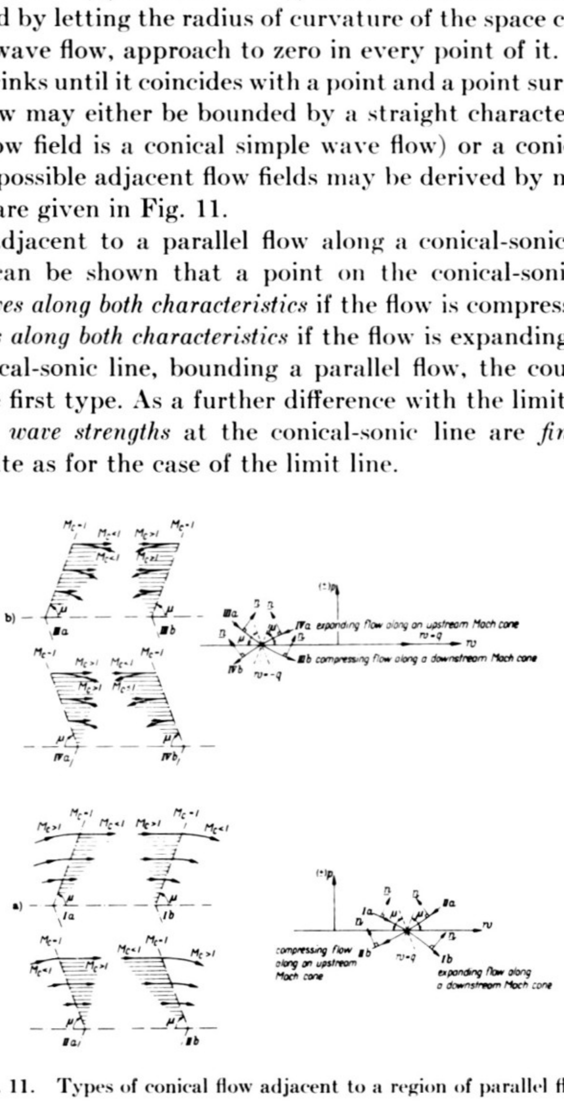

Fig. 11. Types of conical flow adjacent to a region of parallel flow.

SUPERSONIC FLOW ON THE EXPANSION SIDE OF A FLAT SWEPT FORWARD ARROW WING WITH SUPERSONIC LEADING EDGES

The supersonic flow around a flat swept forward arrow wing with supersonic leading edges was discussed before by Bulakh.[4] In order to add to this discussion and as an example how knowledge of the local properties of the hodograph transformation may be used for the qualitative description of a conical flow field, the flow on the expansion side will now be investigated. Consider a flat wing, as sketched in Fig. 12, of which the leading edges are swept forward, such that the velocity component normal to a leading edge is supersonic, placed at an angle of attack in a uniform stream and use the coordinate system sketched in Fig. 12. A picture of the flow on the expansion side of the wing is given in Fig. 13 in such a way that the flow field on the unit sphere is projected centrally

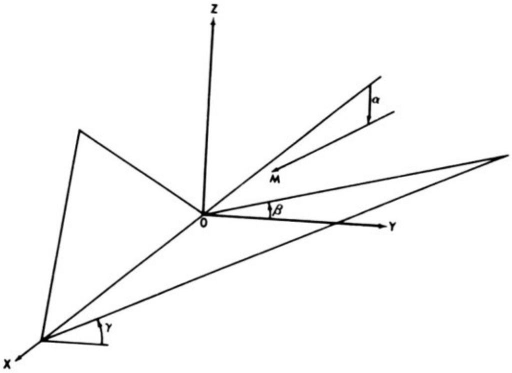

Fig. 12. Geometry of a flat swept-forward arrow wing with supersonic leading edges.

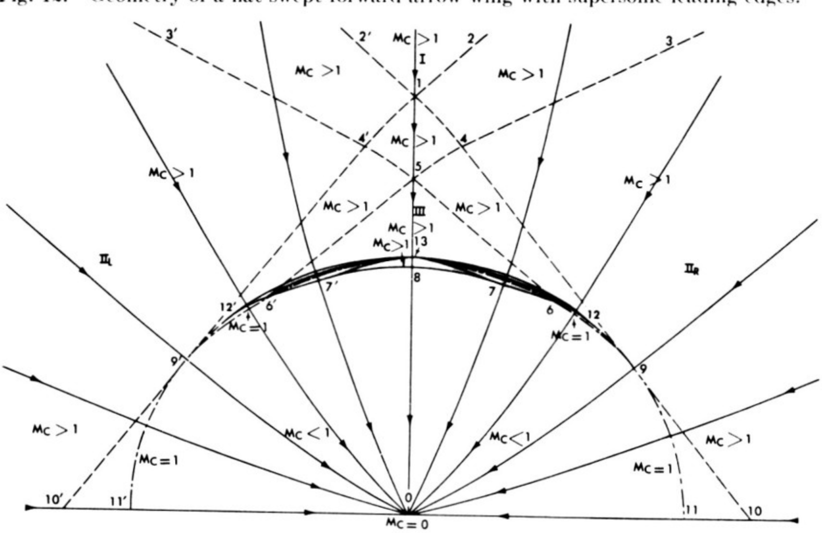

Fig. 13. Flow pattern on the expansion side of a flat swept-forward arrow wing with supersonic leading edges.

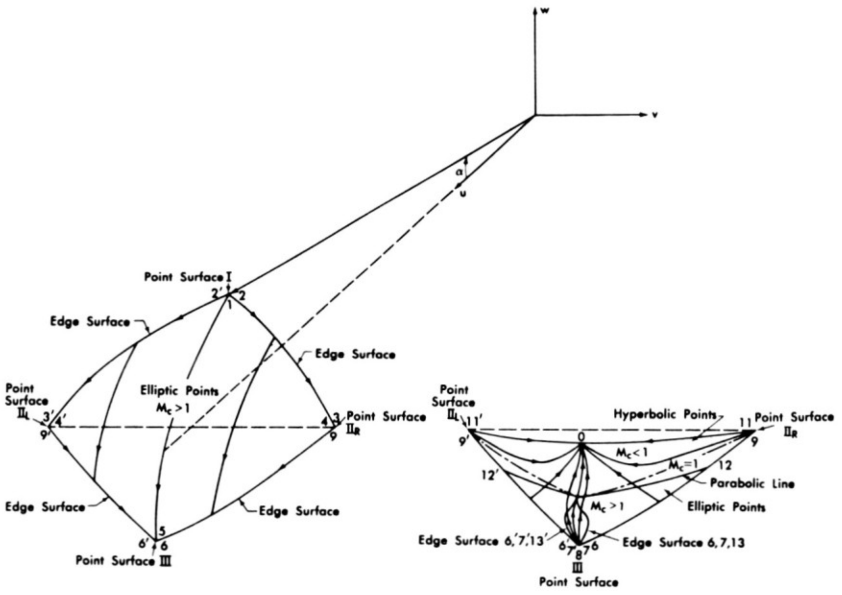

Fig. 14. Hodograph surface of the flow of Fig. 13.

from the center of the conical flow onto a plane, downstream of 0, normal to the center line of the wing. In Fig. 14, the corresponding surface in the hodograph space is sketched. The parallel flow in the undisturbed stream I is bounded by the straight conical characteristics, 1-2 and 1-2′, going through the intersection points of the leading edges with the unit sphere. These points of intersection are not shown in Fig. 13. Obviously, since the parallel flow I is mapped onto a point surface I, where $\rho_1 = \rho_2 = 0$, both families of conical characteristics do not carry elementary contributions. Adjacent to the straight characteristic 1-2 (and 1-2′), there will be a conical simple wave flow, the straight characteristics of which are centered at the leading edge intersection with the unit sphere. Such an intersection point is a conical limit line of the first type, degenerated to a point, and generates expansion waves, which travel downstream along the straight characteristics in the region 1-2-3-4 (1-2′-3′-4′). The flow expands until the velocity vector is parallel to the wing surface and another region of parallel flow II occurs near the wing surface, in the hodograph space given by the point surfaces II_L and II_R. The conical simple wave flows are given in the hodograph space by the edge surfaces I II_R and I II_L, which coincide with plane curves lying in planes perpendicular to the respective leading edges. On the edge surfaces $\rho_1 > 0$, since the flow is expanding, except for the lines of parabolic edge point ($\rho_1 \to \infty$) which are the images of the centerpoints.

The expansion waves of both simple wave regions interact in the characteristic quadrangle 1-4-5-4′ and the flow is fully expanding in this region and mapped onto a region of elliptic points. Downstream of the characteristic 4′-5 and (4-5), the expansion waves continue in the simple wave region 4′-5-9′-6′ (4-5-9-6), which is mapped onto the edge surface II_L III (II_R III), which coincides with a space curve, so that the straight characteristics are not centered. The region downstream of the straight characteristics 5-6 and 5′-6′ is a region of parallel flow III, mapped in the hodograph space onto the point surface III, which coincides with the endpoint of a velocity vector, pointing downwards with respect to the wing surface.

The region where the influence of the wing apex is noticeable is now confined to the region downstream of the curve 11-9-6-7-8-7′-6′-9′-11′ of which the corresponding part of the hodograph surface is drawn in Fig. 14 as a separate surface, which actually should be attached to the part of the surface on the left side along the curve II_L III and II_R III. In fact, virtually the same boundary value problem is obtained as for the supersonic delta wing. If, namely, Fig. 13 is turned around over 90° to the right, the left hand side of the flow becomes similar to the right side of the flow on the expansion side of the delta wing. The line 1-5-8-0 is equivalent to the delta wing surface and the line 10′-11′-0 to the line of symmetry.

The parallel flows II_R and II_L are bounded by the characteristics 3-4 and 3′-4′, the wing surface and the downstream parts of the characteristics through 4 and 4′, respectively. Such a downstream part continues to the point, where this straight characteristic is tangent to the circular conical-sonic line in the region of parallel flow. The continuation of the characteristic beyond this point, 9-10 (and 9′-10′) is an upstream part of it and from the point 9 on, the border of the parallel flow II is given by the conical-sonic line 9-11 (and 9′-11′), being the envelope of straight characteristics in the parallel flow.

Downstream of the line 9′-11′ (9-11) the flow is expanding and a continuous transition of the type I*b* of Fig. 11 to conical-subsonic flow occurs. On the downstream side the simple wave flow region 4′-5-6′-9′ (4-5-6-9) is bordered by the downstream curved characteristic 9′-6′ (9-6) through the point 9′ (9). Along this line the influence of the apex of the wing, experienced along the conical-sonic line 9′-11′ (9-11), propagates in downstream direction from 9′(9) into the direction of 6′ (6). The region of parallel flow III is bounded downstream by the straight characteristic 6′-7′ (6-7), which is the continuation of the characteristic 9′-6′ (9-6), and for lower angles of attack by the conical-sonic line 7-8-7′. Downstream of characteristic 9′-6′ (9-6) the flow is still expanding in the neighborhood of 9′, this expansion becoming less and less if one moves along 9′-6′ in the direction of 6′; then starts to develop into compressing flow until the deceleration becomes infinite at some point 12′ (12). Approaching from the other side it may be shown that the conical-sonic line 7-8-7′ is a limit line, since downstream of it the flow must be compressing in order to turn the downward pointing velocity in the direction of the wing center line. The type of transition III*b* of Fig. 11 to conical-supersonic flow thus occurs. The continuation of the conical-sonic line, the characteristic 7′-6′ (7-6) and the part 6′-12′ (6-12) of characteristic 6′-9′ (6-9) are then a conical limit line of the second type. The reversed flow is bounded downstream by a limit line of the first type 12-13-12′, where the flow once more reverses and after crossing the conical-sonic line 9-13-9′ enters the conical-subsonic region around the conical stagnation point 0, where all conical streamlines converge to.

The waves that arrive at the characteristic 9′-6′ (9-6) may be split up into a group that arrives at 9′-12′ (9-12) and a group arriving at 12′-6′ (12-6). The first group of expansion waves crosses the characteristic 9′-12′ (9-12) without change in its wave strength and is reflected as compression waves at the conical-sonic line 9′-13′ (9-13) along some part near the point 9′ (9). Between 9′-12′ (9-12) and the conical-sonic line a region of partially expanding flow occurs near 9′ (9) and a region of partially compressing flow occurs further away from 9′ (9), both regions being mapped onto a region of hyperbolic points on the hodograph surface. Eventually, the reflected characteristics become tangent to the limit line of the first type 12′-13-12 and the compression waves are absorbed by the limit line along a part from the point 12′ (12) on. The expansion waves of the other group are, at the limit line of the second type 12′-6 (12-6) reflected into the sheet of reversed flow as compression waves of equal strength. From the sheet of reversed flow they are at the limit line of the first type 12′-13-12 reflected again into the sheet of the unreversed flow as expansion waves with equal strength. Another reflection occurs at the conical-sonic line 9′-13 (9-13), where the expansion waves are reflected again as compression waves which travel further downstream in order to be absorbed by the limit line 12′-13-12, by reaching points on the limit line along the characteristics which are tangent to the limit line. Thus, all the waves reaching the characteristic 9′-6′ (9-6) eventually are absorbed by the limit line over the length 12′-13 (12-13). The whole region 9′-12′-13-6′-9′ (9-12-13-6-9) in the unreversed flow is either partially expanding or partially compressing and mapped onto a region of hyperbolic points. In the reversed flow, the characteristics in the region 12′-6′-13 (12-6-13), which belong to the same family as the characteristic 6′-13 (6-13), carry compression waves. The remaining wave systems in the sheet of reversed flow is governed by the conical-sonic line 7′-8-7, which generates compression waves of finite strength along both families of characteristics. Waves, generated along 7′-8-7 and traveling to the left (right) enter the compression simple wave region 6′-7′-13 (6-7-13) across the characteristic 7′-13 (7-13) and proceed along the straight characteristics. These simple wave flow regions are mapped onto the edge surfaces 6′, 7′, 13′ and 6, 7, 13 in the hodograph space. They then continue as compression waves in the region 12′-6′-13 (12-6-13) in order to be absorbed by the limit line 12′-13 (12-13) by approaching the limit line along characteristics tangent to it in the sheet of reversed flow. The wave, generated at point 7′ (7) is absorbed at 12′ (12) and that generated at point 7 (7′) is absorbed at 13 (13). In the region 12′-6′-13 (12-6-13), the flow is fully compressing and the region is mapped onto a region of elliptic points on the hodograph surface. In the simple wave region 6′-7′-13 (6-7-13) the curved characteristics, starting at 6′-7′ (6-7), do not carry elementary contributions and all converge to one point 13′ (13). Since in the region 7′-8-7-13 both characteristics carry compression waves, the flow therein is fully compressing flow and is mapped onto a region of elliptic points on the hodograph surface.

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